Chapter 4

Triangle Congruence by ASA and AAS

What You’ll Learn

• To prove two triangles congruent using the ASA Postulate and the AAS Theorem

...And Why

To prove that the two sides of a lacrosse goal are congruent triangles, as in Example 2

Check Skills You’ll Need

In \( \triangle HKJ \), which side is included between the given pair of angles?
1. \( \angle J \) and \( \angle H \)
2. \( \angle H \) and \( \angle K \)

In \( \triangle NLK \), which angle is included between the given pair of sides?
3. \( \angle L \) and \( \angle L \)
4. \( \angle N \) and \( \angle N \)

Give a reason to justify each statement.
5. \( P \equiv P \)
6. \( \angle A \equiv \angle D \)

If 2 \( \angle \) of a \( \triangle \) are \( \equiv \) to 2 \( \angle \) of another \( \triangle \), the third \( \angle \) are \( \equiv \).

1. Plan

Objectives

1. To prove two triangles congruent using the ASA Postulate and the AAS Theorem

Examples

1. Using ASA
2. Real-World Connection
3. Planning a Proof
4. Writing a Proof

Math Background

ASA is presented in this lesson as a postulate, but it could be established as a theorem (whose proof requires constructing congruent segments) that follows from the SAS postulate, much as SSS also could be established as a theorem that follows from the SAS Postulate. The proof of the AAS Theorem follows from the ASA Postulate and the Triangle Angle-Sum Theorem.

Lesson Planning and Resources

See p. 196E for a list of the resources that support this lesson.

PowerPoint

Bell Ringer Practice

Check Skills You’ll Need

For intervention, direct students to:

Using the SAS and SSS Postulates
Lesson 4-2: Example 2
Extra Skills, Word Problems, Proof Practice, Ch. 4

Proving Triangles Congruent
Lesson 4-1: Example 4
Extra Skills, Word Problems, Proof Practice, Ch. 4

1. Using the ASA Postulate and the AAS Theorem

In Lesson 4-2 you learned that two triangles are congruent if
two pairs of sides are congruent and the included angles are congruent (SAS).
The Activity Lab on page 204 suggests that two triangles are also congruent if
two pairs of angles are congruent and the included sides are congruent (ASA).

Key Concepts

Postulate 4-3 Angle-Side-Angle (ASA) Postulate

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.

\( \triangle HGB \equiv \triangle NKP \)

1. EXAMPLE Using ASA

Multiple Choice Which triangle is congruent to \( \triangle CAT \) by the ASA Postulate?

- \( \triangle DOG \)
- \( \triangle INF \)
- \( \triangle GDO \)
- \( \triangle FNI \)

\( \angle C \equiv \angle G, \angle CA \equiv \angle GD, \) and \( \angle A \equiv \angle D \). \( \triangle CAT \equiv \triangle GDO \) by ASA. Choice C is correct.

Quick Check

Can you conclude that \( \triangle INF \) is congruent to either of the other two triangles? Explain. No; the \( \equiv \) side is not the included side.

Differentiated Instruction

Special Needs
Have students mark congruent sides and angles as shown to distinguish between AAS and ASA.

Below Level
Students may substitute specific measures for the congruent angles in the flow proof of the AAS Theorem to see the dependence on the ASA Postulate before proceeding to the general proof.

Learning Styles
- Visual
- Verbal
Here is how you can use the ASA Postulate in a proof.

**Example**

### Real-World Connection

Lacrosse Study what you are given and what you are to prove about the lacrosse goal. Then write a proof that uses ASA.

**Given:** \( \angle CAB \cong \angle DAE, \overline{AB} \cong \overline{AE}, \angle ABC \) and \( \angle AED \) are right angles.

**Prove:** \( \triangle ABC \cong \triangle AED \)

**Proof:** \( \triangle ABC \cong \triangle AED \) because all right angles are congruent.

You are given that \( \overline{AB} \cong \overline{AE} \) and \( \angle CAB \cong \angle DAE \).

Thus, \( \triangle ABC \cong \triangle AED \) by ASA.

### Quick Check

Write a proof.

**Given:** \( \overline{NM} \cong \overline{NP}, \angle M \cong \angle P \)

**Prove:** \( \triangle NML \cong \triangle NPO \)

It is given that \( \overline{NM} \cong \overline{NP} \) and \( \angle M \cong \angle P \). \( \triangle NML \cong \triangle NPO \) because vert. \( \angle s \) are \( \cong \). \( \triangle NML \cong \triangle NPO \) by ASA.

You can use the ASA Postulate to prove the Angle-Angle-Side Congruence Theorem. A flow proof is shown below.

### Key Concepts

#### Theorem 4-2

**Angle-Angle-Side (AAS) Theorem**

If two angles and a nonincluded side of one triangle are congruent to two angles and the corresponding nonincluded side of another triangle, then the triangles are congruent.

\( \triangle CDM \cong \triangle XGT \)

### Proof of the Angle-Angle-Side Theorem

**Given:** \( \angle A \cong \angle X, \angle B \cong \angle Y, \overline{BC} \cong \overline{YZ} \)

**Prove:** \( \triangle ABC \cong \triangle XYZ \)

<table>
<thead>
<tr>
<th>( \angle A \cong \angle X )</th>
<th>( \angle C \cong \angle Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Given</strong></td>
<td><strong>Given</strong></td>
</tr>
<tr>
<td>( \angle B \cong \angle Y )</td>
<td>( \angle B \cong \angle Y )</td>
</tr>
</tbody>
</table>

If 2 \( \triangle s \) of one \( \triangle \) are \( \cong \) to 2 \( \triangle s \) of another \( \triangle \), then the 3rd \( \triangle s \) are \( \cong \).

\( \triangle ABC \cong \triangle XYZ \)

**ASA Postulate**

A proof starts in your head with a plan. It may help to jot down your plan. You will find this especially helpful as proofs become more complex. As you become skilled at proof, you will find that a good plan looks a lot like a paragraph proof.

### Differentiated Instruction

#### Advanced Learners

After students read the ASA Postulate and the AAS Theorem, ask: If you could use only one in the remainder of this course, which would you choose and why?

**learning style: verbal**

#### English Language Learners

Point out that AAS, or Angle-Angle-Side Theorem, is a theorem and not a postulate. A postulate is an accepted statement of fact, but theorems are conjectures that are proven.

**learning style: verbal**
Lesson 4-3
Triangle Congruence by ASA and AAS

Planning a Proof
Study what you are given and what you are to prove. Then plan a proof that uses AAS.

Given:
\[ \angle S \cong \angle Q, \]
\[ \overline{RP} \text{ bisects } \angle SRQ. \]

Prove:
\[ \triangle SRP \cong \triangle QRP \]

Plan:
\[ \triangle SRP \cong \triangle QRP \text{ by AAS} \]

- The second statement is true by the Reflexive Property of Congruence.

Use the plan from Example 3 and write a proof.

Writing a Proof
Study what you are given and what you are to prove. Then write a proof that uses AAS.

Given:
\[ \overline{XR} \parallel \overline{TQ}, \overline{XR} \text{ bisects } \overline{QT}. \]

Prove:
\[ \triangle XMQ \cong \triangle RMT \]

Plan:
\[ \overline{XR} \parallel \overline{TQ} \text{ gives two pairs of congruent alternate interior angles. } \overline{XR} \text{ bisects } \overline{QT} \]
\[ \text{means } \overline{QM} \cong \overline{TM}. \text{ Use AAS.} \]

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. \overline{XR} \parallel \overline{TQ}</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. \angle Q \cong \angle T, \angle X \cong \angle R</td>
<td>2. ?</td>
</tr>
<tr>
<td>3. \overline{XR} \text{ bisects } \overline{QT}.</td>
<td>3. Given</td>
</tr>
<tr>
<td>4. \overline{QM} \cong \overline{TM}.</td>
<td>4. Definition of segment bisector</td>
</tr>
<tr>
<td>5. \triangle XMQ \cong \triangle RMT</td>
<td>5. AAS</td>
</tr>
</tbody>
</table>

Quick Check
a. Supply the reason that justifies Step 2. If \parallel \text{ lines, then alt. int. } \angle \text{ are } \cong. 

b. Critical Thinking Explain how you could prove \triangle XMQ \cong \triangle RMT by ASA. \triangle XMQ \cong \triangle RMT \text{ because vert. } \triangle \text{ are } \cong.

Closure
Explain why the letters of ASA and AAS are written in a different order. ASA compares triangles in which \parallel \text{ sides are between the two pairs of } \cong \text{ angles, and AAS compares triangles in which } \cong \text{ sides are not between pairs of } \cong \text{ angles.}

EXERCISES
For more exercises, see Extra Skill, Word Problem, and Proof Practice.

Practice and Problem Solving

Practice by Example
Example 1 (page 213)

Name two triangles that are congruent by the ASA Postulate.

1. \triangle PQR \cong \triangle VXW
2. \triangle ACB \cong \triangle EFD

Answer each question without drawing the triangle.
3. Which side is included between \angle R and \angle S in \triangle RST? \overline{RS}
4. Which angles include \overline{NO} in \triangle NOM? \angle N and \angle O

Lesson 4-3  Triangle Congruence by ASA and AAS 215
5. **Developing Proof**  Complete the proof by filling in the blanks.

   **Given:** \(\angle LKM \cong \angle JKM, \)
   \(\angle LMK \cong \angle JMK\)
   **Prove:** \(\triangle LKM \cong \triangle JKM\)
   **Proof:** \(\triangle LKM \cong \triangle JKM\)
   \(\angle LMK \cong \angle JMK\) are
given. \(KM \cong KM\) by the a. **Postulate.**
   Reflexive
   \(\triangle LKM \cong \triangle JKM\) by the b. **Postulate.**

6. **Given:** \(\triangle ABC \cong \triangle DAC \)
   \(AC \perp BD\)
   **Prove:** \(\triangle ABC \cong \triangle ADC\)
    **Plan:** \(\triangle ABC \cong \triangle ADC\) by AAS if \(\angle ABC \cong \angle ADC\)
   \(\angle ACB \cong \angle ACD\), and \(\overline{AC} \cong \overline{AC}\)
   \(\overline{AC} \cong \overline{AC}\) because all c. **angles are congruent.** right
   \(\overline{AC} \cong \overline{AC}\) by the d. **Property of Congruence.**
   Reflexive

7. **Given:** \(\overline{QR} \parallel \overline{ST}\)
   **Prove:** \(\triangle QRT \cong \triangle TSQ\)
   **Proof:** \(\triangle QRT \cong \triangle TSQ\)
   **See back of book.**

8. **Developing Proof**  Complete the proof plan by filling in the blanks.

   **Given:** \(\angle UWT\) and \(\angle UWV\) are right angles,
   \(\angle T \cong \angle V\).
   **Prove:** \(\triangle UWT \cong \triangle UWV\)
   **Plan:** \(\triangle UWT \cong \triangle UWV\) by AAS if \(\angle T \cong \angle V\),
   \(\angle UWT \cong \angle UWV\), and \(\overline{UW} \cong \overline{UW}\)
   \(\angle UWT \cong \angle UWV\) because all c. **angles are congruent.** right
   \(\overline{UW} \cong \overline{UW}\) by the d. **Property of Congruence.**
   Reflexive

9. Use your plan from Exercise 8 and write a proof.  **See back of book.**

10. **Developing Proof**  Complete the proof by filling in the blanks.

   **Given:** \(\angle N \cong \angle S, \text{line } \ell \text{ bisects } \overline{TQ} \text{ at } Q.\)
   **Prove:** \(\triangle NQT \cong \triangle SQR\)
   **Proof:** \(\triangle NQT \cong \triangle SQR\)
   **Statements**
   1. \(\angle N \cong \angle S\)
   2. \(\angle NQT \cong \angle SQR\)
   3. \(\ell \text{ bisects } \overline{TQ} \text{ at } Q\)
   4. \(\text{Definition of bisect}\)
   5. \(\triangle NQT \cong \triangle SQR\)
   **Reasons**
   1. Given
   2. **Vert. are \(\cong.**
   3. **Given
   4. **Definition of bisect
   5. **AAS

11. **Given:** \(\angle V \cong \angle Y\),
    \(\overline{WZ} \text{ bisects } \angle VWY.\)
    **Prove:** \(\triangle VWZ \cong \triangle YWZ\)
    **Proof:** \(\triangle VWZ \cong \triangle YWZ\)
    **11–12. See back of book.**

12. **Given:** \(\overline{PQ} \perp \overline{ON}, \overline{RS} \perp \overline{ON}\)
    \(T\) is the midpoint of \(\overline{PR}.\)
    **Prove:** \(\triangle PQT \cong \triangle RST\)
    **Proof:** \(\triangle PQT \cong \triangle RST\)
Write a congruence statement for each pair of triangles. Name the postulate or theorem that justifies your statement.

13. \( \triangle PMO \cong \triangle NMO; \) ASA
14. \( \triangle UTS \cong \triangle RST; \) AAS
15. \( \triangle ZYV \cong \triangle WYV; \) AAS

16. **Multiple Choice** For the triangles shown,
   \( \angle D \cong \angle T, \angle E \cong \angle U, \) and \( \overline{EO} \cong \overline{UX}. \)
   Which of the following statements is true? \( \text{D} \)
   \( \text{A} \) \( \triangle TUX \cong \triangle DOE \)
   \( \text{B} \) \( \triangle UTX \cong \triangle DOE \)
   \( \text{C} \) \( \triangle TUX \cong \triangle DOE \)
   \( \text{D} \) \( \triangle TUX \cong \triangle DOE \)

17. **Writing** Anita says that you can rewrite any proof that uses the AAS Theorem as a proof that uses the ASA Postulate. Do you agree with Anita? Explain.

18. \( \angle E \cong \angle I \) and \( \overline{FE} \cong \overline{GI}. \) What else must you know to prove \( \triangle FDE \cong \triangle GHI \)
   by AAS? By ASA? \( \triangle FDE \cong \triangle GHI; \triangle DFE \cong \triangle HGI \)

19. **Reasoning** Can you prove the triangles at the right congruent using ASA or AAS? Justify your answer.
   No; also need one pair of corresp. sides. \( \equiv \)

20. \( \triangle MON \cong \triangle QOP \) by AAS; \( \angle MON \) and \( \angle QOP \) are \( \equiv \) vert. \( \angle. \)
21. \( \triangle FGJ \cong \triangle HJG \) by AAS since \( \angle FGJ \cong \angle HJG \)
   because when lines are \( \parallel \), then alt. \( \angle. \) are \( \equiv \) and \( \overline{GJ} \equiv \overline{GJ} \)
   by the Reflexive Prop. of \( \equiv. \)
22. \( \triangle AEB \cong \triangle BDC \) by ASA, since \( \angle EAB \cong \angle DBC \)
   because \( \parallel \) lines have \( \equiv \) corr. \( \angle. \)
23. \( \triangle BDH \cong \triangle FDH \) by ASA since \( \angle BDH \cong \angle FDH \)
   \( \) by def. of \( \angle. \) bis.
   \( \) and \( \overline{DH} \equiv \overline{DH} \)
   by the Reflexive Prop. of \( \equiv. \)
24. **Constructions** Using a straightedge, draw a triangle. Label it \( \triangle JKL. \)
   Construct \( \triangle MNP \cong \triangle JKL \) so you know that the triangles are congruent by ASA.
   See back of book.
25. **Proof** \( \triangle ABC \cong \triangle CDA \)
   See back of book.
26. **Reasoning** If possible, draw two noncongruent triangles that have two pairs of congruent angles and one pair of congruent sides. If this is not possible, explain why. See left.

**Alternative Method**
**Exercise 10** Challenge students to find a second way to complete the proof. Ask: **Suppose that you did not know the Vertical Angles Theorem. How could you find another pair of congruent angles?**

Because \( \angle N \equiv \angle S, \overline{NT} \parallel \overline{RS} \) by the Converse of the Alt. Int. Angles Thm., so \( \angle T \equiv \angle R \) because they are alt. int. angles. \( \overline{QT} \equiv \overline{QR} \)
by definition of a bisector, and \( \triangle NQT \equiv \triangle SQR \) by AAS.

**Error Prevention!**
**Exercise 11** Students sometimes think an angle bisector also bisects the opposite side or is a perpendicular bisector. Ask:
From the Given, can you conclude \( \angle PQS \equiv \angle RSQ \) or \( \overline{PS} \equiv \overline{QR}? \) No Review how angle bisectors, segment bisectors, and perpendicular bisectors are different.

**Exercise 16** Point out that one triangle must be flipped over in order for the triangles to match exactly.

**Visual Learners**
**Exercise 22** Because the parallel segments terminate at the transversal \( \overline{AC}, \) students may have difficulty spotting corresponding angles. Have them copy just the parallel segments \( \overline{AE} \) and \( \overline{BD} \) and the transversal \( \overline{AC} \) to help them find the corresponding angles more easily.

**Connection to Discrete Math**
**Exercise 31** There are 20 ways because \( g_3 = 6 \cdot 5 \cdot 4 = 20. \) To list the sets of three efficiently, suggest that students rename the congruence statements 1, 2, 3, 4, 5, and 6.
27. **Open-Ended** Draw a triangle. Draw a second triangle that shares a common side with the first one and is congruent to it. **Check students’ work.**

b. Think about how you drew your second triangle. What postulate or theorem did you use to make the second triangle congruent to the first one? most likely ASA

Use the figure at the right. Name as many pairs of congruent triangles as you can for the information given.

28. \(ABCD\) is a parallelogram. See margin.

29. \(ABCD\) is a rectangle.

30. **Reasoning** \(\triangle JKL \cong \triangle MNP\).

What additional information about \(KQ\) and \(NR\) will allow you to conclude that \(\triangle JKQ \cong \triangle MNR\)? Explain.

They are \(\angle\) bisectors; ASA.

31. **Probability** Here are six congruence statements about the triangles at the right. \(13\)

\(\angle A \cong \angle X\)
\(\angle B \cong \angle Y\)
\(\angle C \cong \angle Z\)

\(AB \cong XY\)
\(AC \cong XZ\)
\(BC \cong YZ\)

There are 20 ways to choose a group of three statements from these six. What is the probability that three statements chosen at random from the six will guarantee that the triangles are congruent?

32. \(\triangle RST\) at the right has \(RS = 5, RT = 9,\) and \(m\angle T = 30\). Show that there is no SSA congruence rule by constructing \(\triangle UVW\) with \(UV = 5, UW = 9,\) and \(m\angle W = 30\), but with \(\triangle UVW \cong \triangle RST\). See left.

33. Which of the following is NOT a method used to prove triangles congruent? 
   A. AAS   B. ASA   C. SAS   D. SSA

34. Suppose \(\overline{RT} \cong \overline{ND}\) and \(\angle R \cong \angle N\). What additional information is needed to prove \(\triangle RTJ \cong \triangle NDF\) by ASA? 
   F. \(\angle T \cong \angle D\)
   G. \(\angle R \cong \angle N\)
   H. \(\angle J \cong \angle D\)
   J. \(\angle T \cong \angle F\)

35. **Extended Response** \(\overline{PQ}\) bisects \(\angle RPS\) and \(\angle RQS\). Justify each answer.
   a. Which pairs of angles, if any, are congruent? 
   b. By what theorem or postulate can you prove that \(\angle PRQ \cong \angle PSQ\)? See margin.
   c. Can the two triangles be proved congruent by ASA? Explain.

Have students explain the four ways they have learned to prove triangles congruent—SSS, SAS, ASA, and AAS—and include a diagram with each method.
In Exercises 37 and 38, decide whether you can use the SSS Postulate or the SAS Postulate to prove the triangles congruent. If so, write the congruence statement and name the postulate. If not, write not possible.

37. \( \triangle OLN \cong \triangle MLN; \) SAS

38. not possible

39. For any \( \triangle ABC, \) which sides are not included between \( \angle A \) and \( \angle B? \)

\( \overline{AC} \) and \( \overline{CB} \)

40. State the theorem or postulate that justifies the statement:
   If \( \angle 1 \cong \angle 3, \) then \( a \parallel b. \)

If corr. \( \angle \) are \( \cong, \) then the lines are \( \parallel. \)

Photography You want to arrange class-trip photos without overlap to make a 2 ft-by-3 ft poster. You collect 3 in.-by-5 in. and 4 in.-by-6 in. photos. What is the greatest number of each type of photo that you can fit on your poster?

41. 3 in.-by-5 in.  \( \boxed{56} \)

42. 4 in.-by-6 in. \( \boxed{36} \)

43. What percent more paper is used for a large photo than a regular photo?

60% more paper

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Use this Checkpoint Quiz to check students' understanding of the skills and concepts of Lessons 4-1 through 4-3.

**Resources**

- Grab & Go
- Checkpoint Quiz 1

**Checkpoint Quiz 1**

Lessons 4-1 through 4-3

1. \( \triangle RST \cong \triangle KLP. \) List the three pairs of congruent corresponding sides and the three pairs of congruent corresponding angles.

\( \overline{RS} \cong \overline{JK}; \overline{ST} \cong \overline{KL}; \overline{RT} \cong \overline{JL}; \angle R \cong \angle J; \angle S \cong \angle K; \angle T \cong \angle L \)

State the postulate or theorem you can use to prove the triangles congruent. If the triangles cannot be proven congruent, write not possible.

2. ASA

3. SSS

4. SAS

5. not possible

6. not possible

7. AAS

Use the information given in the diagram.
Tell why each statement is true.

8. \( \angle H \cong \angle K \) If \( \parallel \) lines, then corr. \( \angle \) are \( \cong. \)

9. \( \triangle HNL \cong \triangle KNJ \) Vert. \( \angle \) are \( \cong. \)

10. \( \triangle HNL \cong \triangle KNJ \) ASA or AAS

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b. Yes; \( \angle JLM \cong \angle KGM \) because they are alt. int. \( \angle \) of \( \parallel \) lines, and \( \angle LMJ \cong \angle GMK \) because vertical \( \angle \) are \( \cong. \) So the \( \angle \) are \( \cong \) by ASA.

c. Yes; if two \( \angle \) of one \( \triangle \) are \( \cong \) to 2 \( \angle \) of another \( \triangle, \) the third \( \angle \) are \( \cong. \)

[3] incorrect \( \angle \) for part b or c, but otherwise correct

[2] correct conclusions but incomplete explanations for parts b and c

[1] at least one part correct