Trapezoids and Kites

What You’ll Learn
• To verify and use properties of trapezoids and kites

...And Why
To find angle measures of trapezoidal windows, as in Example 2

New Vocabulary
• base angles of a trapezoid

Properties of Trapezoids and Kites

The parallel sides of a trapezoid are its bases. The nonparallel sides are its legs. Two angles that share a base of a trapezoid are base angles of the trapezoid.

The following theorem is about each pair of base angles. You will be asked to prove it in Exercise 38.

The bases of a trapezoid are parallel. Therefore the two angles that share a leg are supplementary. This fact and Theorem 6-15 allow you to solve problems involving the angles of a trapezoid.

Key Concepts

The base angles of an isosceles trapezoid are congruent.

EXAMPLE 1 Finding Angle Measures in Trapezoids

\[ A\text{BCD is an isosceles trapezoid and } m\angle B = 102. \]

Find \( m\angle A, m\angle C, \) and \( m\angle D. \)

Two angles that share a leg are supplementary.

\[ m\angle A + m\angle B = 180 \]

Substitute.

\[ m\angle A + 102 = 180 \]

Subtract 102 from each side.

\[ m\angle A = 78 \]

By Theorem 6-15, \( m\angle C = m\angle B = 102 \) and \( m\angle D = m\angle A = 78. \)
In the isosceles trapezoid, \( m \angle S = 70 \).

Find \( m \angle P, m \angle Q, \) and \( m \angle R \).

110, 110, 70

**Example** Real-World Connection

**Architecture** The second ring of the ceiling shown at the left is made from congruent isosceles trapezoids that create the illusion of circles. What are the measures of the base angles of these trapezoids?

Each trapezoid is part of an isosceles triangle whose base angles are the acute base angles of the trapezoid. The isosceles triangle has a vertex angle that is half as large as one of the 20 angles at the center of the ceiling.

The measure of each angle at the center of the ceiling is \( \frac{360}{20} \) or 18.

The measure of \( \angle 1 \) is \( \frac{180}{2} \), or 9.

The measure of each acute base angle is \( \frac{180}{2} - 9 \), or 85.5.

The measure of each obtuse base angle is \( 180 - 85.5 \), or 94.5.

A glass ceiling like the one above has 18 angles meeting at the center instead of 20.

What are the measures of the base angles of the trapezoids in its second ring? 85, 95

Like the diagonals of parallelograms, the diagonals of an isosceles trapezoid have a special property.

**Theorem 6-16**

The diagonals of an isosceles trapezoid are congruent.

**Proof of Theorem 6-16**

**Given:** Isosceles trapezoid \( ABCD \) with \( AB \cong DC \)

**Prove:** \( AC \cong DB \)

It is given that \( AB \cong DC \). Because the base angles of an isosceles trapezoid are congruent, \( \angle ABC \cong \angle DCB \). By the Reflexive Property of Congruence, \( BC \cong BC \). Then, by the SAS Postulate, \( \triangle ABC \cong \triangle DCB \). Therefore, \( AC \cong DB \) by CPCTC.

Another special quadrilateral that is not a parallelogram is a kite. The diagonals of a kite, like the diagonals of a rhombus, are perpendicular. A proof of this for a kite (next page) is quite like its proof for a rhombus (at the top of page 330).
Proof of Theorem 6-17

Given: Kite $RSTW$ with $TS \cong TW$ and $RS \cong RW$

Prove: $TR \perp SW$

Both $T$ and $R$ are equidistant from $S$ and $W$.

By the Converse of the Perpendicular Bisector Theorem, $T$ and $R$ lie on the perpendicular bisector of $SW$. Since there is exactly one line through any two points (Postulate 1-1), $T$ must be the perpendicular bisector of $SW$.

Therefore, $TR \perp SW$.

You can use Theorem 6-17 to find angle measures in kites.

### Finding Angle Measures in Kites

1. Find $m\angle 1$, $m\angle 2$, and $m\angle 3$ in the kite.

   \[ m\angle 1 = 72, m\angle 2 = 90, m\angle 3 = 18 \]

### Additional Examples

3. Find $m\angle 1$, $m\angle 2$, and $m\angle 3$ in the kite.

   \[ m\angle 1 = 72, m\angle 2 = 90, m\angle 3 = 18 \]

### Resources

- Daily Notetaking Guide 6-5
- Daily Notetaking Guide 6-5—Adapted Instruction

### Closure

Draw and label an isosceles trapezoid, a (convex) kite, and their diagonals. Then write congruence statements for all pairs of triangles that you can prove congruent. Students should find three pairs of congruent triangles for each figure.
7. **Design** Each patio umbrella is made of eight panels that are congruent isosceles triangles with parallel stripes. A sample panel is shown at the right. The vertex angle of the panel measures 42.

7a. **isos. trapezoids**

a. Classify the quadrilaterals shown as blue stripes on the panel.

b. Find the measures of the quadrilaterals' interior angles. 69, 69, 111, 111

**Example 3** (page 338)

Find the measures of the numbered angles in each kite.

10. 108, 108

11. 22°, 90, 68

12. 45°, 45, 45°

13. 54°, 90°

14. 38°, 1, 53°, 5

15. 36°, 2, 71°, 5

16. 34°, 1, 46°, 5

**Exercise 10** Discuss ways to prove \( m \angle 1 = m \angle 2 \).

**Exercise 38** Have students work together to write this proof. Even with the Plan, the proof is complex and worthy of class discussion.

17. **Open-Ended** Sketch two kites that are not congruent, but with the diagonals of one congruent to the diagonals of the other. **See margin.**

18. The perimeter of a kite is 66 cm. The length of one of its sides is 3 cm less than twice the length of another. Find the length of each side of the kite.

19. **Critical Thinking** If \( KLMN \) is an isosceles trapezoid, is it possible for \( KN \) to bisect \( \angle LMN \) and \( \angle LKN \)? Explain. **See margin.**

**Algebra** Find the value of the variable in each isosceles trapezoid.

20. \( 2x \) \( 12 \)

21. \( 3x \) \( 15 \)

22. \( 3x + 15 \) \( 15 \)

23. \( 2x - 1 \) \( 3 \)

24. \( x + 1 \) \( 4 \)

25. \( x + 5 \) \( 1 \)

\( TV = 2x - 1 \)

\( US = x + 2 \)

\( SU = x + 1 \)

\( TR = 2x - 3 \)

\( QS = x + 5 \)

\( RP = 3x + 3 \)

**Homework Quick Check**
To check students' understanding of key skills and concepts, go over Exercises 6, 14, 29, 37, 38.

**Diversity**
**Exercise 7** The word *umbrella* comes from a Latin word meaning “shaded area or shadow,” suggesting protection against the rain or sun. Ask whether students know the word for umbrella in other languages. For example, the Spanish word *paraguas* literally means “for water,” and a *sombrilla* is a parasol.

**Exercise 10** Discuss ways to prove \( m \angle 1 = m \angle 2 \).

**Exercise 38** Have students work together to write this proof. Even with the Plan, the proof is complex and worthy of class discussion.

19. **Explanations may vary.** Sample: **If both \( \angle s \) are bisected, then this combined with \( KM \equiv KM \) by the Reflexive Prop. means \( \triangle KLM \equiv \triangle KN \) by SAS. By CPCTC, opp. \( \angle s \equiv \angle s \). \( \angle L \) and \( \angle N \) are opp., but \( KLMN \) is isos., both pairs of base \( \angle s \) are also \( \equiv \). By the Trans. Prop., all 4 angles are \( \equiv \), so \( KLMN \) must be a rect.**

or a square. This contradicts what is given, so \( KM \) cannot bisect \( \angle LMN \) and \( \angle LKN \).
4. Assess & Reteach

**Lesson Quiz**

Use isosceles trapezoid $ABCD$ for Exercises 1 and 2.

1. If $m \angle A = 45$, find $m \angle B$, $m \angle C$, and $m \angle D$. $m \angle B = 45$, $m \angle C = m \angle D = 135$
2. If $AC = 3x - 16$ and $BD = 10x - 86$, find $x$. $x = 10$

Use kite $GHIJ$ for Exercises 3–6.

3. Find $m \angle 1$. $90$
4. Find $m \angle 2$. $9$
5. Find $m \angle 3$. $81$
6. Find $m \angle 4$. $40$

**Alternative Assessment**

Have students work in pairs to write answers to the following questions:
- How are a kite and a rhombus similar? How are they different?
- How are an isosceles trapezoid and a rectangle similar? How are they different?

34. No; if two consecutive $\triangle$ are suppl., then another pair must be also because one pair of opp. $\triangle$ is $\equiv$. Therefore, the opp. $\triangle$ would be $\equiv$, which means the figure would be a $\text{a}$ and not a kite.

36. No; if two consecutive $\triangle$ are compl., then the kite would be concave. Rhombuses and squares would be kites since opp. sides can be $\equiv$ also.

37. Writing A kite is sometimes defined as a quadrilateral with two pairs of consecutive sides congruent. Compare this to the definition you learned in Lesson 6–1. Are parallelograms, trapezoids, rhombuses, rectangles, or squares special kinds of kites according to the changed definition? Explain. See margin.

38. Developing Proof The plan suggests a proof of Theorem 6–15. Write a proof that follows the plan. See back of book.

Given: Isosceles trapezoid $ABCD$ with $\overline{AB} \equiv \overline{DC}$
Prove: $\angle B \equiv \angle C$ and $\angle BAD \equiv \angle D$

Plan: Begin by drawing $\overline{AE} \parallel \overline{DC}$ to form parallelogram $AEDC$ so that $\overline{AE} \equiv \overline{DC} \equiv \overline{AB}$.
$\angle B \equiv \angle C$ because $\angle B \equiv \angle 1$ and $\angle 1 \equiv \angle C$. Also, $\angle BAD \equiv \angle D$ because they are supplements of the congruent angles, $\angle B$ and $\angle C$.

Write a proof. Use the given figure with additional lines as needed.

39. Given: Isosceles trapezoid $TRAP$ with $\overline{TR} \equiv \overline{PA}$
Prove: $\angle RTA \equiv \angle APR$ See margin.

40. Given: Isosceles trapezoid $TRAP$ with $\overline{TR} \equiv \overline{PA}$; $\overline{HI}$ is the perpendicular bisector of $\overline{RA}$ at $B$ and $\overline{TP}$ at $I$. See margin.
Prove: $\overline{HT}$ is the perpendicular bisector of $\overline{TP}$.

For a trapezoid, consider the segment joining the midpoints of the two given segments. How are its length and the lengths of the two parallel sides of the trapezoid related? Justify your answer.

41. the two nonparallel sides
42. the diagonals

43. $\overline{BN}$ is the perpendicular bisector of $\overline{AC}$ at $N$. Describe the set of points, $D$, for which $ABCD$ is a kite. See above left.

44. Prove that the angles formed by the noncongruent sides of a kite are congruent. ($\text{Hint:}$ Draw a diagonal of the kite.) See back of book.

**Algebra** Find the value(s) of the variable(s) in each kite.

26. $x = 2x$
27. $(3x + 5)^{\circ} = (2y - 40)^{\circ}$
28. $x = 35, y = 30$

29. Bridge Design A quadrilateral is formed by the beams of the bridge at the left.

Isosc. trapezoid; all the large rt. $\triangle$ appear to be $\equiv$.

30. Find the measures of the other interior angles of the quadrilateral. 112, 68, 68

31. Critical Thinking Can two angles of a kite be as follows? Explain. 31–33. See below left.

- opposite and acute
- consecutive and supplementary
- opposite and supplementary
- consecutive and supplementary

33. Yes; if two consecutive $\angle$ are compl., then another pair must be also because one pair of opp. $\angle$ is $\equiv$. Therefore, the opp. $\angle$ would be $\equiv$, which means the figure would be a $\triangle$ and not a kite.

36. No; if two consecutive $\triangle$ are compl., then the kite would be concave. Rhombuses and squares would be kites since opp. sides can be $\equiv$ also.

37. Writing A kite is sometimes defined as a quadrilateral with two pairs of consecutive sides congruent. Compare this to the definition you learned in Lesson 6–1. Are parallelograms, trapezoids, rhombuses, rectangles, or squares special kinds of kites according to the changed definition? Explain. See margin.

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For a trapezoid, consider the segment joining the midpoints of the two given segments. How are its length and the lengths of the two parallel sides of the trapezoid related? Justify your answer.

41. the two nonparallel sides
42. the diagonals

43. $\overline{BN}$ is the perpendicular bisector of $\overline{AC}$ at $N$. Describe the set of points, $D$, for which $ABCD$ is a kite. See above left.

44. Prove that the angles formed by the noncongruent sides of a kite are congruent. ($\text{Hint:}$ Draw a diagonal of the kite.) See back of book.
45. Which statement is true for every trapezoid?  
A. Exactly two sides are congruent.  
B. Exactly two sides are parallel.  
C. Opposite angles are supplementary.  
D. The diagonals bisect each other.

46. Which statement is true for every kite?  
F. Opposite sides are congruent.  
G. At least two sides are parallel.  
H. Opposite angles are supplementary.  
J. The diagonals are perpendicular.

47. Two consecutive angles of a trapezoid are right angles. Three of the following statements about the trapezoid could be true. Which statement CANNOT be true?  
A. The two right angles are base angles.  
B. The diagonals are not congruent.  
C. Two of the sides are congruent.  
D. No two sides are congruent.

48. Quadrilateral $EFGH$ is a kite. What is the value of $x$?  
H. 15  
G. 70  
F. 85  
J. 160

49. In the trapezoid at the right, $BE = 2x - 8$, $DE = x - 4$, and $AC = x + 2$.  
a. Write and solve an equation for $x$.  
b. Find the length of each diagonal.

50. Diagonal $RB$ of kite $RHBW$ forms an equilateral triangle with two of the sides. $m \angle BWR = 40$. Draw and label a diagram showing the diagonal and the measures of all the angles. Which angles of the kite are largest?  
See margin.