Proportions in Triangles

What You’ll Learn
• To use the Side-Splitter Theorem
• To use the Triangle-Angle-Bisector Theorem

...And Why
To design a sail, as in Example 2

Check Skills You’ll Need

The two triangles in each diagram are similar. Find the value of \(x\) in each.

1. \(\frac{84\text{ cm}}{x} = \frac{30\text{ cm}}{15\text{ cm}}\)
2. \(\frac{x}{3\text{ mm}} = \frac{6\text{ mm}}{11\text{ mm}}\)
3. \(\frac{12\text{ in.}}{x} = \frac{5\text{ in.}}{7\text{ in.}}\)
4. \(\frac{x}{3\text{ ft}} = \frac{7.5\text{ ft}}{6\text{ ft}}\)

Using the Side-Splitter Theorem

You can use similar triangles to prove the following theorem.

Key Concepts

\textbf{Theorem 7-4} Side-Splitter Theorem

If a line is parallel to one side of a triangle and intersects the other two sides, then it divides those sides proportionally.

\textbf{Proof of Theorem 7-4}

Given: \(\triangle QXY\) with \(RS \parallel YX\)

Prove: \(\frac{XR}{RQ} = \frac{YS}{SQ}\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
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</thead>
<tbody>
<tr>
<td>1. (RS \parallel YX)</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. (\angle 1 \cong \angle 3, \angle 2 \cong \angle 4)</td>
<td>2. If lines are (\parallel), then corr. (\triangle) are (\cong).</td>
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<tr>
<td>3. (\triangle QXY \sim \triangle QRS)</td>
<td>3. (\triangle) Postulate</td>
</tr>
<tr>
<td>4. (\frac{XR}{RQ} = \frac{YS}{SQ})</td>
<td>4. Corr. sides of (\sim) (\triangle) are proportional.</td>
</tr>
<tr>
<td>5. (XR = XR + RQ, YQ = YS + SQ)</td>
<td>5. Segment Addition Postulate</td>
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<tr>
<td>6. (\frac{XR + RQ}{RQ} = \frac{YS + SQ}{SQ})</td>
<td>6. Substitute.</td>
</tr>
<tr>
<td>7. (\frac{XR}{RQ} = \frac{YS}{SQ})</td>
<td>7. A Property of Proportions</td>
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Differentiated Instruction

Special Needs
Have students draw three parallel lines cut by two transversals. Students use a ruler to measure the segments intercepted on the transversals and observe they are proportional, and not congruent.

Below Level
Review the properties of proportions, especially the Cross-Product Property, before students read the proof of Theorem 7-4 and work through the examples.

Learning style: tactile

Learning style: verbal
EXAMPLE Using the Side-Splitter Theorem

Grided Response Find the value of $x$.

\[
\frac{TS}{ST} = \frac{TU}{TV} \quad \text{Side-Splitter Theorem}
\]

\[
\frac{15}{ST} = \frac{5}{16}
\]

Substitute.

\[
x = \frac{5}{10} \cdot 16
\]

Solve for $x$.

\[
x = 8
\]

Quick Check 1. Use the Side-Splitter Theorem to find the value of $x$. 

2. The following corollary to the Side Splitter Theorem says that parallel lines divide all transversals proportionally. You will prove this corollary in Exercise 35.

Example 2 Real-World Connection

Sail Making Sail makers sometimes use a computer to create a pattern for a sail. After they cut out the panels of the sail, they sew them together to form the sail.

The edges of the panels in the sail at the right are parallel. Find the lengths $x$ and $y$.

\[
\frac{2}{x} = \frac{1.7}{1.7}
\]

Side-Splitter Theorem

\[
x = 2
\]

Corollary to the Side-Splitter Theorem

\[
\frac{\frac{3}{2}}{2} = \frac{y}{1.7}
\]

\[
y = 2.55
\]

Quick Check 2. Solve for $x$ and $y$.

\[
x = \frac{22}{13}; y = 28.6
\]
You can use the Side-Splitter Theorem to prove the following relationship.

### Key Concepts

#### Theorem 7-5  Triangle-Angle-Bisector Theorem

If a ray bisects an angle of a triangle, then it divides the opposite side into two segments that are proportional to the other two sides of the triangle.

### Proof of Theorem 7-5

**Given:** $\triangle ABC$, $\overline{AD}$ bisects $\angle CAB$.

**Prove:** $\frac{CD}{DB} = \frac{CA}{BA}$

**Proof:** By the Side-Splitter Theorem, $\frac{AD}{DB} = \frac{CA}{BA}$

By the Corresponding Angles Postulate, $\angle 3 \cong \angle 1$.

Since $\overline{AD}$ bisects $\angle CAB$, $\angle 1 \cong \angle 2$. By the Alternate Interior Angles Theorem, $\angle 2 \cong \angle 4$. Using the Transitive Property of Congruence, you know that $\angle 3 \cong \angle 4$.

By the Converse of the Isosceles Triangle Theorem, $BA = AF$. Substituting $BA$ for $AF$, $\frac{CD}{DB} = \frac{CA}{AF}$.

### Practice and Problem Solving

#### Additional Examples

**Example 1**

1. Find the value of $x$.

**Algebra** Find the value of $x$.

**PS**  $\triangle ABC$, $\overline{AM}$ bisects $\angle BAC$. Find $x$ and $y$.

**Resources**

- Daily Notetaking Guide 7-5
- Daily Notetaking Guide 7-5—Adapted Instruction

#### Error Prevention

Some students looking at the diagram for this example may think they can use the proportions from Lesson 7-4 that apply only to right triangles. Ask: What must be true to apply the theorems and corollaries from Lesson 7-4? The triangle must be a right triangle with an altitude to the hypotenuse.

#### EXERCISES

For more exercises, see Extra Skill, Word Problem, and Proof Practice.

#### Practice and Problem Solving

**A Practice by Example**

**Algebra** Solve for $x$.

**Example 1** (page 399)

1. $x = 3.6$, $y = 4.8$

2. $x + 4 = 9$

3. $13 - x = 5.2$
Use the figure at the right to complete each proportion.

4. \( \frac{a}{b} = \frac{d}{c} \)
5. \( \frac{b}{a} = \frac{f}{c} \)
6. \( \frac{a}{d} = \frac{b}{e} \)
7. \( \frac{a}{b} + \frac{b}{c} = \frac{c}{d} \)

Algebra Solve for \( x \).

8. 9. 10.

Use the figure at the right to complete each proportion.

11. \( \frac{KS}{KS} = \frac{RI}{RI} \)
12. \( \frac{KL}{KL} = \frac{M}{M} \)
13. \( \frac{KL}{KL} = \frac{PM}{PM} \)

Algebra Solve for \( x \).

14. 3.6
15. 40
16. 4.8

Urban Design In Washington, D.C., 17th, 18th, 19th, and 20th Streets are parallel streets that intersect Pennsylvania Avenue and I Street.

25. How long (to the nearest foot) is Pennsylvania Avenue between 19th Street and 18th Street? 559 ft
26. How long (to the nearest foot) is Pennsylvania Avenue between 18th Street and 17th Street? 671 ft

27. The sides of a triangle are 5 cm, 12 cm, and 13 cm long. Find the lengths, to the nearest tenth, of the segments into which each angle divides the opposite side. 2.4 cm and 2.6 cm; 3.3 cm and 8.7 cm; 3.8 cm and 9.2 cm

28. Open-Ended In a triangle, the bisector of an angle divides the opposite side into two segments with lengths 6 cm and 9 cm. How long could the other two sides of the triangle be? (Caution: Make sure the three sides satisfy the Triangle Inequality Theorem.) Answers may vary. Sample: 9 cm and 13.5 cm
Chapter 7

30a. [Diagram of a triangle with a bisector]

30. Critical Thinking Sharell draws \( \triangle ABC \). She finds that the bisector of \( \angle C \) bisects the opposite side.
   a. Sketch \( \triangle ABC \) and the bisector. \( \text{See left.} \)
   b. Writing What type of triangle is \( \triangle ABC \)? Explain your reasoning.

31. Algebra Solve for \( x \).
   31. \[ \begin{align*}
   4x + 8 &= 10x - 4 \\
   4x &= 6x - 12 \\
   6x &= 20 \\
   x &= \frac{20}{6} \\
   x &= \frac{10}{3}
   \end{align*} \]

32. [Diagram of a triangle with segments]

33. [Diagram of a triangle with segments]

34. Developing Proof Copy and complete this two-column proof of the Converse of the Side-Splitter Theorem: If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

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<td>a. ( \text{Given} )</td>
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<td>2. ( \frac{XR + RQ}{RQ} = \frac{YS + SQ}{SQ} )</td>
<td>b. ( \text{Prop. of Proportions} )</td>
</tr>
<tr>
<td>3. ( \frac{XQ}{RQ} = \frac{YS}{SQ} )</td>
<td>c. ( \text{Segment Add. Post.} )</td>
</tr>
<tr>
<td>4. ( \angle Q \equiv \angle Q )</td>
<td>d. ( \text{Reflexive Prop. of ( \equiv )} )</td>
</tr>
<tr>
<td>5. ( \triangle XQY \sim \triangle RQS )</td>
<td>e. ( \text{SAS ( \sim ) Thm.} )</td>
</tr>
<tr>
<td>6. ( \angle 1 \equiv \angle 2 )</td>
<td>f. ( \text{Corr. ( \angle ) of ( \sim ) are ( \equiv )} )</td>
</tr>
<tr>
<td>7. ( RS \parallel XY )</td>
<td>g. ( \text{If corr. ( \angle ) are ( \equiv ), lines ( \parallel )} )</td>
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35. Proof 35. Follow the steps below. Write a proof of the Corollary to the Side-Splitter Theorem found on page 399.

Given: \( AW \parallel RX \parallel CY \)

Prove: \( \frac{AB}{BC} = \frac{WX}{XY} \)

Begin by drawing \( WC \), intersecting \( RX \) at point \( Z \).
   a. Apply the Side-Splitter Theorem to \( \triangle ACW \): \( \frac{WZ}{ZC} = \frac{WX}{XY} \)
   b. Apply the Side-Splitter Theorem to \( \triangle CWY \): \( \frac{WZ}{ZC} = \frac{WX}{XY} \)
   c. Substitute to prove the corollary: \( \frac{AB}{BC} = \frac{WX}{XY} \)

Determine whether the red segments are parallel. Explain each answer. You can use the theorem proved in Exercise 34. \( \text{See margin.} \)

36. Yes; since \( \frac{6}{10} = \frac{9}{15} \), the segments are \( \parallel \) by the Converse of the Side-Splitter Thm.

37. No; \( \frac{20}{12} \neq \frac{24}{10} \).

38. Yes; since \( \frac{15}{12} = \frac{20}{16} \), the segments are \( \parallel \) by the Converse of the Side-Splitter Thm.
39. **Oil Spills** Describe how you could use the figure at the right to find the length of the oil spill indirectly. What measurements and calculations would you use? See margin.

40. An angle bisector of a triangle divides the opposite side of the triangle into segments 5 cm and 3 cm long. A second side of the triangle is 7.5 cm long. Find all possible lengths for the third side of the triangle. 4.5 cm or 12.5 cm

**Geometry in 3 Dimensions** In the figure at the right, \( \overrightarrow{FG} \parallel \overrightarrow{AB}, \overrightarrow{GH} \parallel \overrightarrow{BC} \). \( AF = 2, FE = 4, \) and \( BG = 3. \)

41. Find \( GE. \) 6

42. If \( EH = 5, \) find \( HC. \) 2.5

43. If \( FG = 3, \) find the perimeter of \( \triangle ABE. \) 19.5

44. One side of a triangle is \( k \) times as long as a second side. The bisector of their angle cuts the third side into two segments. Prove that one of those segments is \( k \) times as long as the other.

45. Use the definition in part (a) to prove the statements in parts (b) and (c).
   a. Write a definition for a midsegment of a parallelogram.
   b. A parallelogram midsegment is parallel to two sides of the parallelogram.
   c. A parallelogram midsegment bisects the diagonals of a parallelogram.

   a–c. See back of book.

46. State the converse of the Triangle-Angle-Bisector Theorem. Give a convincing argument that the converse is true or a counterexample to prove that it is false. See back of book.

47. In \( \triangle ABC, \) the bisectors of \( \angle A, \angle B, \) and \( \angle C \) cut the opposite sides into lengths \( a_1 \) and \( b_1, b_2, \) and \( c_1 \) and \( c_2, \) respectively, labeled in this order counter-clockwise around \( \triangle ABC. \) Find the perimeter of \( \triangle ABC \) for each of the following.

   a. \( a_1 = \frac{5}{7}, b_1 = \frac{15}{4} \) 14
   b. \( a_1 = \frac{10}{7}, b_1 = \frac{10}{7}, c_1 = \frac{8}{7} \) 11

**Test Prep**

### Multiple Choice

50. [2] \( a + \frac{1}{3} = \frac{20}{35}; 35n + 35 = 560; \) \( n = 15 \)
   [1] correct proportion solved incorrectly

### Short Response

50. What is \( n? \) Show your work. See left.

### Extended Response

51. The bisectors of an angle of a triangle divide the opposite side of the triangle into segments 4 cm and 5 cm long. A second side of the triangle is 6 cm long.
   a. Draw two diagrams you can use to find the two possible different lengths for the third side. a–b. See back of book.
   b. Use each diagram in part (a) to write a proportion. Solve for each possible length of the third side of the triangle. Show your work.

39. Measure \( \overline{AC}, \overline{CE}, \) and \( \overline{BD}. \)
   Use the Side-Splitter Thm.
   Write the proportion. \( \frac{AC}{CE} = \frac{AB}{BD} \) and solve for \( AB. \)
Refer to the figure to complete each proportion.

52. \[ \frac{m}{n} = \frac{b}{c} \]
53. \[ \frac{m}{n} = \frac{b}{c} \]
54. \[ \frac{m}{n} = \frac{b}{c} \]
55. \[ \frac{m}{n} = \frac{b}{c} \]

Lesson 6-4

Algebra

RSTV is a rectangle. Find the lengths of the diagonals \( RT \) and \( SV \).

56. \( RT = SV = 38 \)
57. \( RT = SV = 37 \)
58. \( RT = 8x - 4, SV = 6x + 9 \)
59. \( RT = 3x + 5, SV = 5x + 4 \)

Lesson 5-3

Find the center of the circle that you can circumscribe about each \( \triangle ABC \).

60. \( \triangle ABC \) \( A \) \((0, 0) \) \((3, -3) \)
61. \( \triangle ABC \) \( A \) \((2, 5) \) \((0, 2) \)
62. \( \triangle ABC \) \( A \) \((-2, 0) \) \((1.5, 2.5) \)

For \( \triangle ABC \), name the point of intersection associated with each set of segments.

63. the medians
64. the angle bisectors
65. the altitudes

Lesson 7-3 through 7-5

Determine whether the triangles are similar. If so, write the similarity statement. Also, write the postulate or theorem that proves they are similar.

1. \( \triangle ABC \sim \triangle XYZ; \ AA \sim \text{Post.} \)
2. \( \triangle WST \sim \triangle HJG; \ SAS \sim \text{Thm.} \)

Algebra

The polygons are similar. Find the value of each variable.

3. \[ x = \frac{3\sqrt{13}}{2}, \ w = 4.5 \]

Algebra

Find the value of each variable.

4. \[ y = 5, \ x = 15 \]
5. \[ x = 7.5 \]
6. \[ 4\sqrt{5} \]
7. \[ x = 17.5 \]
8. \[ x = 3.6 \]
9. \[ x = 10 \]
10. \[ x = 16 \]