If two figures are congruent, there is a transformation that maps one onto the other. If no reflection is involved, then the figures are either translation or rotation images of each other.

### Recognizing the Transformation

The two figures are congruent. Is one figure a translation image of the other, a rotation image, or neither? Explain.

- The orientations of these congruent figures do not appear to be opposite, so one is a translation image or a rotation image of the other.
- Clearly, it’s not a translation image, so it must be a rotation image.

Any translation or rotation can be expressed as the composition of two reflections.

### Example 1: Recognizing the Transformation

The two figures are congruent. Is one figure a translation image of the other, a rotation image, or neither? Explain.

Neither; the figures do not have the same orientation.

### Theorems

- **Theorem 9-1:** A translation or rotation is a composition of two reflections.
- **Theorem 9-2:** Any translation or rotation can be expressed as the composition of two reflections.
- **Theorem 9-3:** The examples that illustrate Theorems 9-2 and 9-3 suggest a proof of Theorem 9-1 (how to find two reflections for a given translation or rotation).
Theorems 9-2 and 9-3 together form the converse of Theorem 9-1.

**Key Concepts**

**Theorem 9-2**
A composition of reflections across two parallel lines is a translation.

**Theorem 9-3**
A composition of reflections across two intersecting lines is a rotation.

**2 EXAMPLE**

**Composition of Reflections Across Parallel Lines**

Find the image of R for a reflection across line \( \ell \) followed by a reflection across line \( m \). Describe the resulting translation.

R is translated the distance and direction shown by the green arrow. The arrow is perpendicular to lines \( \ell \) and \( m \) with length equal to twice the distance from \( \ell \) to \( m \).

1. Draw lines \( \ell \) and \( m \) as shown above. Draw R between \( \ell \) and \( m \). Find the image of R for a reflection across line \( \ell \) and then across line \( m \). Describe the resulting translation. **See back of book.**

**3 EXAMPLE**

**Composition of Reflections in Intersecting Lines**

Lines \( a \) and \( b \) intersect in point \( C \) and form acute \( \angle 1 \) with measure 35. Find the image of R for a reflection across line \( a \) and then a reflection across line \( b \). Describe the resulting rotation.

R rotates clockwise through the angle shown by the green arrow. The center of rotation is \( C \) and the measure of the angle is twice \( m \angle 1 \), or 70.

1. Repeat Example 3, but begin with R in a different position. **See back of book.**

**Lessons 9-6**

**Compositions of Reflections** 507

**Guided Instruction**

**Error Prevention**

Students may think that each R should look like a translation of the original R. Have them use paper folding to see why the orientation of the second R must be different from the first and third Rs.

**Additional Examples**

1. Judging by appearances, is one figure a translation image or a rotation image of the other? Explain.

2. Find the image of the figure for a reflection across line \( \ell \) and then across line \( m \).

3. The letter D is reflected across line \( x \) and then across line \( y \). Describe the resulting rotation.

**Advanced Learners**

After Example 2, have students find the image of R reflected across line \( \ell \) and then across line \( m \), when \( \ell \perp m \), and develop a theorem for this composition of reflections.

**English Language Learners**

Watch for students who confuse the vocabulary terms. In Example 1, make sure students understand the transformation of \( S \) is not a translation and the transformation of \( H \) is not a reflection.
Two plane figures A and B can be congruent with opposite orientations. Reflect A and you get a figure A’ that has the same orientation as B. Thus, B is a translation or rotation image of A’. By Theorem 9-1, two reflections map A’ to B. The net result is that three reflections map A to B.

This is summarized in what is sometimes called the Fundamental Theorem of Isometries.

If two figures are congruent and have opposite orientations (but are not simply reflections of each other), then there is a slide and a reflection that will map one onto the other. A is the composition of a glide (translation) and a reflection across a line parallel to the direction of translation.

**Finding a Glide Reflection Image**

**Coordinate Geometry** Find the image of \( \triangle T.E.X \) for a glide reflection where the translation is \((x, y) \rightarrow (x, y - 5)\) and the reflection line is \(x = 0\).

A computer can translate an image and then reflect it, or vice versa. The two rabbit images are glide reflection images of each other.

**Quick Check**

3. Use \( \triangle T.E.X \) from Example 4 above.
   a. Find the image of \( \triangle T.E.X \) under a glide reflection where the translation is \((x, y) \rightarrow (x + 1, y)\) and the reflection line is \(y = -2\). See above.
   b. Critical Thinking Would the result of part (a) be the same if you reflected \( \triangle T.E.X \) first, and then translated it? Explain. Yes; if you reflected it and then moved it right, the result would be the same.

You can map one of any two congruent figures onto the other by a single reflection, translation, rotation, or glide reflection. Thus, you are able to classify any isometry.
Classifying Isometries

Each figure is an isometry image of the figure at the left. Tell whether their orientations are the same or opposite. Then classify the isometry.

- a. Opposite; a reflection
- b. Opposite; a glide reflection
- c. Same; a translation
- d. Same; a rotation

Classify the isometry.

The two figures in each pair are congruent. Is one figure a translation image of the other, a rotation image, or neither? Explain.

1. Rotation
2. Translation
3. Neither; the figures do not have the same orientation.

Find the image of each letter for a reflection across line \( \ell \) and then a reflection across line \( m \). Describe the resulting translation or rotation.

4. \( F \) is translated down twice the distance between \( \ell \) and \( m \).
Connection to Algebra
Exercises 12, 13 Help students discover that the image of \((x, y)\) is \((y, x)\) in the reflection line \(y = x\) and is \((-y, -x)\) in the reflection line \(y = -x\).

Tactile Learners
Exercises 16–23 Suggest that students trace each original figure and manipulate their tracings to help them understand how the transformations were made.

Exercise 26 If students select answer choice A, they most likely read \(x = -2\) incorrectly as \(y = -2\).

Auditory Learners
Exercise 27 Ask several volunteers to read their explanations to the rest of the class and answer questions about the math terminology they used.

Exercises 31–34 A kaleidoscope produces repeated reflections in intersecting mirrors. Consequently, the images are reflected isosceles triangles.

Exercise 45 Encourage students to provide several descriptions to help them realize that the glide and reflection are not unique, although the lines of reflection must be parallel.

Example 4 (page 508)
Find the glide reflection image of \(\triangle PNB\) for the given translation and reflection line. 

Example 5 (page 509)
16. \(\triangle SC\); \(\text{opposite; reflection}\)
17. \(\square SJ\); \(\text{opposite; glide reflection}\)
18. \(\triangle 227\); \(\text{same; translation}\)
19. \(\square 227\); \(\text{same; rotation}\)
20. \(\triangle 227\); \(\text{same; rotation}\)
21. \(\square 227\); \(\text{same; translation}\)
22. \(\triangle 227\); \(\text{opposite; reflection}\)
23. \(\square 227\); \(\text{opposite; glide reflection}\)
24. glide reflection; \((x, y) \rightarrow (x - 2, y - 2), \text{refl. in } y = x - 1\)
25. rotation; \(180^\circ\) about the pt. \((0, \frac{1}{2})\)

27. Odd isometries can be expressed as the composition of an odd number of reflections. Even isometries are the composition of an even number of reflections.

Example 29 (page 509)
Yes; a rotation of \(x^o\) followed by a rotation of \(y^o\) is equivalent to a rotation of \((x + y)^o\).

Example 46 (page 510)
If \(XY\) is reflected in line \(l\), then \(l\) is the \(\perp\) bis. of \(XX'\) and \(YY'\), so \(XX' \parallel YY'\) and \(XX'YY'\) is an isosce. trap. Therefore \(\triangle XYZ\) is an isosce. trap. Therefore \(\triangle XYZ\) is an isosce. trap.

Multiple Choice
28. \(\square\) a translation \((x, y) \rightarrow (x, y - 3)\) followed by a reflection across \(x = -2\)
29. \(\square\) a rotation of \(180^\circ\) about the origin
30. \(\square\) a reflection across \(y = \frac{1}{2}\)
31. \(\square\) a reflection across the y-axis followed by a \(180^\circ\) rotation about the origin

Writing
Reflections and glide reflections are \textit{odd isometries}, while translations and rotations are \textit{even isometries}. Use what you learned in this lesson to explain why these categories make sense. \textit{See left.}

Open-Ended
Draw \(\triangle ABC\). Then, describe a reflection, a translation, a rotation, and a glide reflection, and draw the image of \(\triangle ABC\) for each transformation.

Check students' work.
29. For center of rotation \(P\), does an \(x^o\) rotation followed by a \(y^o\) rotation give the same image as a \(y^o\) rotation followed by an \(x^o\) rotation? Explain. \textit{See margin.}
30. Does an \(x^o\) rotation about a point \(P\) followed by a reflection in a line \(\ell\) give the same image as a reflection in \(\ell\) followed by an \(x^o\) rotation about \(P\)? Explain. \textit{No; explanations may vary.}

29. Yes; a rotation of \(x^o\) followed by a rotation of \(y^o\) is equivalent to a rotation of \((x + y)^o\).
Kaleidoscopes The vibrant images of a kaleidoscope are produced by compositions of reflections in intersecting mirrors. Determine the angle between the mirrors in each kaleidoscope image.

31. 60°  32. 60°

33. 51°

34. 30°

Identify each mapping as a reflection, translation, rotation, or glide reflection. Find the reflection line, translation rule, center and angle of rotation, or glide translation and reflection line.

35. AB→EDC  36. EDC→PQM

37. MNP→EDC  38. MNJ→EDC

39. PQM→JLM  40. MN→EDC

41. JLM→MNJ  42. PQM→KIN

43. KIN→ABC  44. HGF→KIN

45. Describe a glide and a reflection that maps the red R to the blue R. See left.

For the given transformation mapping $\overline{XY}$ to $\overline{X'Y'}$, give a convincing argument why $\overline{XY} \cong \overline{X'Y'}$. 46–48. See margin p. 510.

46. a reflection  47. a translation  48. a rotation

49. The definition states that a glide reflection is the composition of a translation and a reflection. Explain why these can occur in either order. See margin.

50. For lines of reflection $r$ and $s$, does a reflection in $r$ followed by a reflection in $s$ give the same image as a reflection in $s$ followed by a reflection in $r$? Explain. See margin.

$P \rightarrow P'(3, -1)$ for the given translation and reflection line. Find the coordinates of $P$.

51. $(x, y) \rightarrow (x - 3, y); y = 2$  52. $(x, y) \rightarrow (x, y - 3); y = 2$

53. $(x, y) \rightarrow (x - 3, y - 3); y = x$  54. $(x, y) \rightarrow (x + 4, y - 4); y = -x$

55. Name the four types of isometries. glide reflection, reflection, rotation, translation

49. Answers may vary. Sample: since a reflection moves a pt. in the direction ⊥ to the translation, the order does not matter.

50. No; explanations may vary. Sample: If (1, 1) is reflected over the line $y = x$ and then the $x$-axis, the image is (1, −1). If the reflections are reversed, the image is (−1, 1).
Chapter 9

Transformations

Alternative Assessment

Have each student use a right scalene triangle preimage to show that two reflections result in either a translation or a rotation, and that three reflections result in either a reflection or a glide reflection.

Students may use compass, straightedge, ruler, and protractor.

Test Prep

Resources

For additional practice with a variety of test item formats:

• Standardized Test Prep, p. 527
• Test-Taking Strategies, p. 522
• Test-Taking Strategies with Transparencies

57. [2] \(V(−5, 2), T(−4, 0), Y(−1, 3)\) glided give \(V′(−2, −1), T′(−1, −3), Y′(2, 0)\). These vertices reflected over \(y = −x\) give \(V′′(1, 2), T′′(3, 1), Y′′(0, −2)\).

[1] incorrect method or answer


b. Suppose pt. \(A\) in \(F\) is 6 units from \(s\). Thus, \(A\) reflected across \(s\) gives \(A′\), \(x\) units right of \(s\). \(A′\) is then \(PQ + x\) units left of \(t\). Thus, \(A′\) reflected across \(t\) gives \(A″\), \(PQ + x\) units right of \(t\). Thus, the total distance travelled is \(PQ + x + PQ + x = 2PQ\).

Mixed Review

Lesson 9-5

Coordinate Geometry A figure has a vertex at \((-2, 7)\). If the figure has the given type of symmetry, state the coordinates of another vertex of the figure.

59. line symmetry about the \(x\)-axis

60. line symmetry about the \(y\)-axis

61. point symmetry about the origin

62. line symmetry about the line \(y = x\)

Lesson 7-4

Algebra Find the value of \(x\).

63. \(8\sqrt{3}\)

64. 10.5

65. 10

Lesson 5-4

Identify the two statements that contradict each other.

66. I. \(\triangle ABC\) is right. I and II

II. \(\triangle ABC\) is equiangular.

III. \(\triangle ABC\) is isosceles.

67. I. In right \(\triangle ABC, \angle B = 90\).

II. In right \(\triangle ABC, \angle A = 80\).

III. In right \(\triangle ABC, \angle C = 90\).

Multiple Choice

55. Find the image of \(P(11, −5)\) for the translation \((x, y) \rightarrow (x − 12, y − 6)\) followed by a reflection in \(x = 0\).

A. \((1, −11)\) B. \((-1, 11)\) C. \((1, 11)\) D. \((-1, −11)\)

56. A reflection in the \(y\)-axis followed by a reflection in the \(x\)-axis does NOT give the same result as which of the following transformations?

H. A reflection in the \(x\)-axis followed by a reflection in the \(y\)-axis

G. A rotation of 180°

F. A rotation of 90° followed by a reflection in the \(x\)-axis

J. A reflection in the line \(y = x\) followed by a reflection in the line \(y = −x\)

Short Response

57. Find the image of \(\triangle VTY\) for the given glide reflection. Show all your steps.

translation: \((x, y) \rightarrow (x + 3, y − 3)\)

reflection line: \(y = −x\) See margin.

Extended Response

58. Copy the diagram with \(s \parallel t\).

a. Draw the image of \(F\) for a composition of two reflections. Reflect first in line \(s\) and then in line \(t\). a-b. See margin.

b. Explain why the resulting image is the same image as found by translating \(F\) in a direction parallel to \(PQ\) through a distance \(2 \cdot PQ\).