Areas of Regular Polygons

What You’ll Learn
• To find the area of a regular polygon

...And Why
To find the area of pieces of honeycomb material used to build boats, as in Example 3

Check Skills You’ll Need
Find the area of each regular polygon. If your answer involves a radical, leave it in simplest radical form.
1. \(25\sqrt{3}\) cm
2. \(10\) ft
3. \(10\) m

Find the perimeter of the regular polygon.
4. a hexagon with sides of 4 in.
5. an octagon with sides of \(2\sqrt{3}\) cm

Lesson Planning and Resources
See p. 530E for a list of the resources that support this lesson.

Lesson 8-2: Examples 1, 3, 4
Extra Skills, Word Problems, Proof Practice, Ch. 8
Finding Area of Parallelograms and Triangles
Lesson 10-1: Examples 1, 3, Extra Skills, Word Problems, Proof Practice, Ch. 10

Vocabulary Tip
The terms radius and apothem (AP uh them) can each refer to either a segment or its length.

More Math Background: p. 530C

Complex Figure
The figure at the right is a regular pentagon with radii and an apothem drawn. Find the measure of each numbered angle.

1. \(m\angle 1 = \frac{360}{5} = 72\)°
2. \(m\angle 2 = \frac{1}{2}m\angle 1 = \frac{1}{2}(72) = 36\)°
3. \(90 + 36 + m\angle 3 = 180\)
   \[m\angle 3 = 54\]

At the right, a portion of a regular octagon has radii and an apothem drawn. Find the measure of each numbered angle.

1. \(m\angle 1 = 45\)°; \(m\angle 2 = 22.5\)°; \(m\angle 3 = 67.5\)°

Below Level L2
While working through Example 1, have students discuss why the five triangles formed by the radii must be congruent.

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Learning style: verbal
Suppose you have a regular \( n \)-gon with side \( s \). The radii divide the figure into \( n \) congruent isosceles triangles. Each isosceles triangle has area equal to \( \frac{1}{2}as \).

Since there are \( n \) congruent triangles, the area of the \( n \)-gon is \( A = n \cdot \frac{1}{2}as \). The perimeter \( p \) of the \( n \)-gon is \( ns \). Substituting \( p \) for \( ns \) results in a formula for the area in terms of \( a \) and \( p \): \( A = \frac{1}{2}ap \).

### Key Concepts

**Theorem 10-6: Area of a Regular Polygon**

The area of a regular polygon is half the product of the apothem and the perimeter.

\[ A = \frac{1}{2}ap \]

### EXAMPLE 2: Finding the Area of a Regular Polygon

Find the area of a regular decagon with a 12.3-in. apothem and 8-in. sides.

Find the perimeter.

\[ p = ns \]
\[ = 10(8) = 80 \text{ in.} \]

Find the area.

\[ A = \frac{1}{2}ap \]
\[ = \frac{1}{2}(12.3)(80) = 492 \]

The regular decagon has area 492 in\(^2\).

### EXAMPLE 3: Boat Racing

Some boats used for racing have bodies made of a honeycomb of regular hexagonal prisms sandwiched between two layers of outer material. At the right is an end of one hexagonal cell. Find its area.

The radii form six 60° angles at the center. You can use a 30°-60°-90° triangle to find the apothem \( a \).

\[ a = 5\sqrt{3} \]
\[ p = ns \]
\[ = 6(10) = 60 \]

Substitute 6 for \( n \) and 10 for \( s \).

Find the area.

\[ A = \frac{1}{2}ap \]
\[ = \frac{1}{2}(5\sqrt{3})(60) \]
\[ = 259.80762 \] (Use a calculator)

The area is about 260 mm\(^2\).

### Quick Check

Find the area of a regular pentagon with 11.6-cm sides and an 8-cm apothem.

232 cm\(^2\)

### EXAMPLE 3: Finding the Area of a Regular Polygon

Find the area of a regular polygon with twenty 12-in. sides and a 37.9-in. apothem.

4548 in\(^2\)

### EXAMPLE 3: Boat Racing

A library is a regular octagon. Each side is 18.0 ft. The radius of the octagon is 23.5 ft. Find the area of the library to the nearest 10 ft.

About 1560 ft\(^2\)

### Resources
- Daily Notetaking Guide 10-3
- Daily Notetaking Guide 10-3—Adapted Instruction

### Closure

Find the area of a regular pentagon with 7.2-ft sides and a 6.1-ft radius.

About 88 ft\(^2\) or 89 ft\(^2\)

### Advanced Learners

After Example 3, have students write formulas for the areas of regular hexagons, one with sides of length \( s \) and the other with apothem of length \( a \).

Learning style: verbal

### English Language Learners

Help students with the term *circumscribe* in “circumscribe a circle about any regular polygon.” Have them break it into its prefix *circum-* which means “around” and its root *scribe* which means “to write.”

Learning style: verbal
Chapter 10

Note how the answer

One of the smallest space satellites ever developed has the shape of

2

pentagon

73 cm

8

Have students

octagon

2192.4 cm

2

Area

2

≠

a. 40 b. 70

≠

= 12.3 m

s

12 in.

dodecagon

2851.8 ft

a

Challenge 36-38

Surface Areas of Prisms and Cylinders

21.

≠

= 27.5 in.,

m

s

= 24.3 cm,

decagon,

2

2475 in.

octagon,

2

22.

≠

= 29.1 ft,

= 14 cm

s

7-gon,

pentagon,

2

35. The apothem is ⊥ to a

side of the pentagon. Two right Δs are formed with the radii of the

pentagon. So the Δs are

≡ by HL. Therefore the

Δ formed by the apothem and radii are

≡ by CPCTC, and the apothem bisects the vertex ∠.
24. **Multiple Choice** The gazebo in the photo is built in the shape of a regular octagon. Each side is 8 ft long, and its apothem is 9.7 ft. What is the area enclosed by the gazebo?

- A. 38.8 ft²
- B. 77.6 ft²
- C. 232.8 ft²
- D. 310.4 ft²

25. The area of a regular polygon is 36 in.². Find the length of a side if the polygon has the given number of sides. Round your answer to the nearest tenth.

   - a. 3 in.
   - b. 6 in.
   - c. 7.8 in.
   - d. **Estimation** Suppose the polygon is a pentagon. What would you expect the length of its side to be? Explain. See left.

26. A portion of a regular decagon has radii and an apothem drawn. Find the measure of each numbered angle. \( \angle 1 = 36; \angle 2 = 18; \angle 3 = 72 \)

27. **Writing** Explain why the radius of a regular polygon is greater than the apothem. See left.

28. **Constructions** Use a compass to construct a circle. 28a–c. See margin.
   a. Construct four perpendicular radii of the circle.
   b. Construct radii that bisect each of the four right angles.
   c. Connect the consecutive points where the radii intersect the circle. What regular polygon have you constructed? **regular octagon**
   d. **Critical Thinking** How can a circle help you construct a regular hexagon? **Construct a 60° angle with the vertex at circle's center**.

29. A regular hexagon has perimeter 120 m. Find its area. 600 \( \sqrt{3} \) m²

30. **Open-Ended** Create a design using equilateral triangles and regular hexagons that have sides of the same length. Find the area of the completed design.

34a. \( b = s; h = \frac{\sqrt{3}}{2} s \)

   \( A = \frac{1}{2} bh \)

   \( A = \frac{1}{2} s \cdot \frac{\sqrt{3}}{2} s \)

   \( A = \frac{1}{4} s^2 \sqrt{3} \)

34b. apothem = \( \frac{s\sqrt{3}}{2} \)

   \( A = \frac{1}{2} ap = \frac{1}{2} \left( \frac{\sqrt{3}}{6} \right) (3s) \)

   \( A = \frac{1}{2} s^2 \sqrt{3} \)

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34a–b. **See left.**

36. For reg. \( n \)-gon \( ABCDE \ldots \), let \( P \) be the intersection of the bisectors of \( \angle ABC \) and \( \angle BCD \). \( BC \equiv CD \). \( \angle BCP \equiv \angle DCP \) and \( CP \equiv CP \), so \( \triangle ABC \equiv \triangle DCP \), and \( \angle CBP \equiv \angle DCP \). By CPCTC. Since \( \triangle BCP \) is half the size of \( \triangle ABC \) and \( \triangle ABC \equiv \triangle CDE \), \( \triangle CDP \) is half the size of \( \triangle CDE \). By a similar argument, \( P \) is on the bisector of each \( \angle \) around the polygon. The smaller \( \triangle \) formed by each of the \( \angle \) bisectors are all \( \approx \). By the Conv. of the Isosc. \( \triangle \) Thm., each of \( \triangle APB, BPC, CDP \), etc., are isosc. with \( AP \equiv BP \equiv CP \), etc.

Thus, \( P \) is equidistant from the polygon’s vertices, so \( P \) is the center of the polygon and the \( \angle \) bis. are radii.
Test Prep

Resources
For additional practice with a variety of test item formats:
- Standardized Test Prep, p. 593
- Test-Taking Strategies, p. 588
- Test-Taking Strategies with Transparencies

Challenge

35. For Example 1 on page 546, write a proof that the apothem bisects the vertex angle of the isosceles triangle formed by the radii. See margin.

36. Prove that the bisectors of the angles of a regular polygon (given congruent sides and angles) are concurrent and that they are, in fact, radii of the polygon. (Hint: For regular n-gon ABCDE... , let P be the intersection of the bisectors of ∠ABC and ∠BCD. Show that DP must be the bisector of ∠CDE.) See margin, p. 581.

37. Coordinate Geometry: A regular octagon with center at the origin and radius 4 is graphed in the coordinate plane.
   a. Since V₂ lies on the line y = x, its x- and y-coordinates are equal. Use the Distance Formula to find the (2.8, 2.8) coordinates of V₂ to the nearest tenth.
   b. Use the coordinates of V₁ and V₂ and the formula A₁ = bh/2 to find the area of ∆V₁OV₂ to the nearest tenth. 5.6 units²
   c. Use your answer to part (b) to find the area of the octagon to the nearest whole number. 45 units²

38. In ∆ABC, ∠C is acute.
   a. Show that the area of ∆ABC = 1/2bh and h = a sin C.
   b. Complete: The area of a triangle is half the product of ______ and the sine of the ______ angle. __two sides; included__
   c. Show that the area of a regular n-gon with radius r is 1/2nr² sin(360°/n). A(each ∆) = 1/2r² sin(360°/n), so A = nr² sin(360°/n).

Test Prep

Multiple Choice
39. What is the area of a regular pentagon whose apothem is 21.5 mm and perimeter is 182 mm? B
   A. 913.6 mm²  B. 2284.1 mm²  C. 3654.6 mm²  D. 4568.2 mm²

40. The area of a regular octagonal garden is 1235.2 yd². The apothem is 19.3 yd. What is the perimeter of the garden? F
   f. 128 yd  g. 154.4 yd  h. 186.6 yd  j. 192 yd

Short Response
41. The radius of a regular hexagonal sandbox is 5 ft. What is the area to the nearest square foot? B
   A. 30 ft²  B. 65 ft²  C. 75 ft²  D. 130 ft²

Extended Response
42. The perimeter of a regular decagon is 220 in. Its radius is 35.6 in.
   a. Explain how to use the given information to find its area.
   b. Find the area. 42a–b. See margin.

43. In regular hexagon ABCDEF, BC = 8√3 ft.
   a. Find the area of ∆BCG.
   b. Find the area of hexagon ABCDEF.
   c. Describe two different methods for finding the area of hexagon ABCDEF. 43a–c. See margin.

42. [2] a. Divide the decagon into 10 = △. Consider one △ with hyp. of 35.6 and leg 11. The apothem can be found using the Pyth. Thm., so (35.6)² – 11² = 1146.36 and leg = 33.9 in.
   b. A = 1/2 (33.9)(220) = 3729 in.²
   [1] incorrect calculation and correct explanation OR correct calculation and no explanation

   b. Area of ABCDEF = 6 • 48√3 = 288√3.
   c. Find the area of one △ and multi.
      by 6 or use the formula for the area of a reg. polygon. [3] appropriate methods, but with one computational error
   [2] incorrect formulas OR no explanation
44. Find the area of a kite with diagonals 8 m and 11.5 m. 46 m²
45. The area of a kite is 150 in.². The length of one diagonal is 10 in. Find the length of the other diagonal. 30 in.
46. The area of a trapezoid is 42 m². The trapezoid has a height of 7 m and one base of 4 m. Find the length of the other base. 8 m

**Lesson 4-4**

Name the pairs of triangles you would have to prove congruent so that the indicated congruences are true by CPCTC.

Given: \( \angle DAB \cong \angle CBA \), \( AD \cong BC \), \( DF \) bisects \( \angle ADB \), \( CG \) bisects \( \angle BCA \).

47. \( \triangle ACD \cong \triangle BDE \) 48. \( \triangle AC \cong \triangle RF \) 49. \( \angle DFA \cong \angle CGB \)
50. a. Biology The size of a jaguar’s territory depends on how much food is available. Where there is a lot of food, such as in a forest, jaguars have circular territories about 3 mi in diameter. Use 3.14 for \( \pi \) to estimate the area of such a region to the nearest tenth. 7.1 mi²

b. Where food is less available, a jaguar may need up to 200 mi². Estimate the radius of this circular territory. about 8 mi

**Checkpoint Quiz 1**

Find the area of each figure.

1. \(84 \text{ in.}^2\)
2. \(112 \text{ cm}^2\)
3. \(40 \text{ m}^2\)

Find the area of each trapezoid, rhombus, or regular polygon. You may leave answers in simplest radical form.

4. \(135 \text{ in.}^2\)
5. \(58.5 \text{ m}^2\)
6. \(72\sqrt{3} \text{ in.}^2\)
7. \(27\sqrt{3} \text{ ft}^2\)
8. \(32 \text{ yd}^2\)
9. \(16\sqrt{3} \text{ in.}^2\)

10. A regular hexagon has a radius of \(3\sqrt{3} \text{ m}\). Find the area. Show your answer in simplest radical form and rounded to the nearest tenth. \(81\frac{3}{2} \text{ m}^2; 70.1 \text{ m}^2\)